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Lois Mansfield *

ABSTRACT

Finite element methods for nonlinear shell analysis are analyzed using both the minimum potential energy and the mixed formulations. Existence and local uniqueness of both the exact solutions and the corresponding finite element solutions are proved. Error bounds, which are of the same order as for the corresponding linear problems, are established.

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I. Introduction

In this paper we analyze finite element methods for nonlinear shell analysis. We consider both the usual minimum potential energy formulation and the mixed formulation using a modified form of the Hellinger-Reissner stationary variational principle. These were considered in the engineering literature by [1] and [9]. Additional references can also be found in these papers. We shall use the nonlinear shallow shell theory for isotropic shells and shall assume Kirchoff's hypothesis. The von Karman theory for plates is obtained in the special case that the curvature is zero.

From the point of view of mathematical finite element analysis, a novel part of nonlinear shell analysis is the fact that one may have non-unique solutions. Typically one considers the load f to be given by λf_0 where λ is a load parameter, and then tracks successive solutions for different (initially increasing) values of the load parameter. Consider in particular the snap-through buckling of a shell (see p. 370-71 of [5]). As the load parameter λ is increased a critical load λ_{cr} which occurs at a limit point is found. The load curve (see figure 1) would then show unloading and perhaps reloading as other elements of resistance are mobilized by the finite deformations. If this resistance is adequate the load may eventually surpass λ_{cr} provided that the material capacity is not exceeded. There is often a sudden transition from the loading to the reloading branches of the curve at λ_{cr} with a corresponding jump in D, (called snap-through buckling) where D is some scalar representation of the displacement.

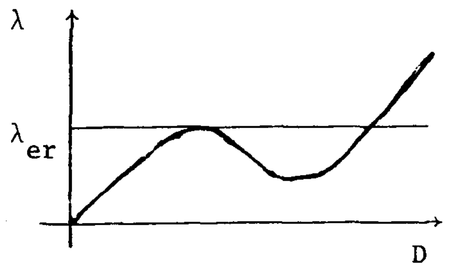

Figure 1.

Our analysis proceeds by using Kantorovich's theorem at successive load steps to prove local existence and uniqueness both of the exact solution and of the finite element solution. We also show that the order of convergence of the finite element approximations is optimal, i.e. the same order as is obtained in the corresponding linear problems. Our analysis is carried out for both the usual minimum potential energy formulation and the mixed formulation using a modified form of the Hellinger-Reissner stationary variational principle.

In our analysis the size of the load increments that can be taken and still satisfy the conditions of Kantorovich's theorem for local existence and uniqueness decreases with increasing displacement in the minimum potential energy method but not in the mixed method. The apparent larger load increments possible in the mixed method seem to be born out in fact in some numerical experiments by the author on a simple one dimensional model problem.

For simplicity we consider the case of the clamped shell throughout. Also, in order to make the equations somewhat simpler, we shall analyze the shallow shell equations rather than the conventional shell equations. Our analysis is no way restricted to shallow shells. In actual computations one would approximate the curvatures $k_{\alpha\beta}$ by piecewise polynomials, or use numerical integration. In this paper we do not consider the effect of such approximations, since the analysis of this effect does not differ from the linear problem, see [3].

II. Minimum Potential Energy Formulation

Let Ω be the shell domain and let $X = H_0^1(\Omega) x H_0^1(\Omega) x H_0^2(\Omega)$. On X we define the norm

$$\| \underline{v} \|_X^2 = \| v_1 \|_{H^1(\Omega)}^2 + \| v_2 \|_{H^1(\Omega)}^2 + \| v_3 \|_{H^2(\Omega)}^2$$

The shell problem can be expressed as: find $\underline{u} = (u_1, u_2, w) \in X$ such that

$$S(\underline{u};f) = \min_{\underline{v} \in X} S(\underline{v};f) = \min_{v \in X} [B(\underline{v},\underline{v}) - 2(f,\underline{v})] \tag{1}$$

where

$$\begin{aligned} B(\underline{v},\underline{v}) = & \int_\Omega A'_{\alpha\beta\gamma\delta} \left(\frac{\partial v_\beta}{\partial x_\alpha} + k_{\alpha\beta} v_3 + \tfrac{1}{2} \frac{\partial v_3}{\partial x_\alpha} \frac{\partial v_3}{\partial x_\beta} \right) \left(\frac{\partial v_\gamma}{\partial x_\delta} + k_{\gamma\delta} v_3 + \tfrac{1}{2} \frac{\partial v_3}{\partial x_\gamma} \frac{\partial v_3}{\partial x_\delta} \right) \\ & + \int_\Omega G'_{\alpha\beta\gamma\delta} \frac{\partial^2 v_3}{\partial x_\alpha \partial x_\beta} \frac{\partial^2 v_3}{\partial x_\gamma \partial x_\delta} , \end{aligned} \tag{2}$$

or equivalently as: find $u \in X$ such that

$$\begin{aligned} \delta S(\underline{u},\underline{v};f) = & \int_\Omega A'_{\alpha\beta\gamma\delta} \left\{ \left(\frac{\partial u_\beta}{\partial x_\alpha} + k_{\alpha\beta} w \right) \left(\frac{\partial v_\gamma}{\partial x_\delta} + k_{\gamma\delta} v_3 \right) \right. \\ & + \tfrac{1}{2} \frac{\partial w}{\partial x_\alpha} \frac{\partial w}{\partial x_\beta} \left(\frac{\partial v_\gamma}{\partial x_\delta} + k_{\gamma\delta} v_3 \right) + \left(\frac{\partial u_\beta}{\partial x_\alpha} + k_{\alpha\beta} w \right) \frac{\partial w}{\partial x_\gamma} \frac{\partial v_3}{\partial x_\delta} \\ & \left. + \tfrac{1}{2} \frac{\partial w}{\partial x_\alpha} \frac{\partial w}{\partial x_\beta} \frac{\partial w}{\partial x_\gamma} \frac{\partial v_3}{\partial x_\delta} \right\} + G'_{\alpha\beta\gamma\delta} \left(\frac{\partial^2 w}{\partial x_\alpha \partial x_\beta} \frac{\partial^2 v_3}{\partial x_\gamma \partial x_\delta} \right) - \int_\Omega f \underline{v} = 0 , \end{aligned} \tag{3}$$

all $\underline{v} \in X$,

where $k_{\alpha\beta}$ denotes the curvature and twist of the shell surface, and $A'_{\alpha\beta\gamma\delta}$ and $G'_{\alpha\beta\gamma\delta}$ are coefficients. Here u_1 and u_2 are the in-plane displacements and w is the transverse displacement. We use the convention that the repeated subscripts α, β, γ, δ indicate implied summations over these subscripts. Throughout we use Greek letters for indices which take their values in the set $\{1, 2\}$.

For the finite element approximation to the solution $\underline{u}$ of (1), choose finite dimensional subspaces $V_h \subset H_0^1(\Omega)$, $W_h \subset H_0^2(\Omega)$, with $X_h = V_h \times W_h$, and find $\underline{u}^h = (u_1^h, u_2^h, w^h) \in X_h$ such that

$$(4) \qquad S(\underline{u}^h;f) = \min_{\underline{v}^h \in X_h} S(\underline{v}^h;f) \quad ,$$

or equivalently, find $\underline{u}^h \in X_h$ such that

$$(5) \qquad \delta S(\underline{u}^h, \underline{v}^h;f) = 0 \quad , \qquad \text{all } \underline{v}^h \in X_h \quad .$$

Our proofs of existence and uniqueness of solutions of (3) and (5) will be based largely on the following lemma, which we state and prove for the exact problem (3) but which applies equally to the approximate problem (5).

Lemma 1. Suppose there exists a solution $\underline{u}_0$ to (3) for $f = \lambda_0 f_0$ such that

$$(6) \qquad \delta^2 S(\underline{u}_0, \underline{v}, \underline{v}) \geq \alpha_0 \, \| \underline{v} \|_X^2$$

Then there exists a $\Delta\lambda$ and an r such that for $f = (\lambda_0 + \Delta\lambda) f_0$, (3) has a unique solution $\underline{u}$ satisfying $\| \underline{u} - \underline{u}_0 \| \leq r$.

Proof. The proof consists of verifying the hypotheses of Kantorovich's theorem for the solution of equations in Banach spaces by Newton's method. Observe that $\delta S(u_0,\underline{v};(\lambda_0+\Delta\lambda)f_0) = \int_\Omega \Delta\lambda f_0\underline{v}$ and thus the solution $\underline{\tilde{u}}$ to

$$(7) \qquad \delta^2 S(\underline{u}_0,\underline{\tilde{u}},\underline{v}) = \delta S(\underline{u}_0,\underline{v};(\lambda_0+\Delta\lambda)f_0) \quad , \qquad \text{all } \underline{v} \in X$$

is bounded by

$$(8) \qquad \frac{\Delta\lambda \, \| f_0 \|}{\alpha_0} \quad .$$

It is straightforward to verify using the inequalities

$$\int_\Omega g_1 g_2 g_3 \leq \| g_1 \|_{L^4(\Omega)} \| g_2 \|_{L^4(\Omega)} \| g_3 \|_{L^2(\Omega)} \quad ,$$

$$\int_\Omega g_1 g_2 g_3 g_4 \leq \| g_1 \|_{L^4(\Omega)} \| g_2 \|_{L^4(\Omega)} \| g_3 \|_{L^4(\Omega)} \| g_4 \|_{L^4(\Omega)} \quad ,$$

and the embedding $L^4(\Omega) \subset H^1(\Omega)$, that

$$(9) \qquad |\delta^2 S(\underline{v}_1,\underline{\hat{v}},\underline{\bar{v}}) - \delta^2 S(\underline{v}_2,\underline{\hat{v}},\underline{\bar{v}})| \leq \frac{M}{2} \| \underline{v}_1-\underline{v}_2 \|_X \| \underline{\hat{v}} \|_X \| \underline{\bar{v}} \|_X$$

for all $\underline{v}_1,\underline{v}_2 \in S = \{\underline{v}: \| \underline{v}-\underline{u}_0 \|_X \leq 2\alpha_0\}$, where M depends linearly on $\| (v_3)_1 + (v_3)_2 \|_X$. By Kantorovich's theorem ([7], Chapter 18, see also [4], p. 143-150), if

$$\frac{\Delta\lambda \, \| f \|_M}{\alpha_0^2} = H_0 \leq \tfrac{1}{2} \quad ,$$

then there exists a unique solution $\underline{u}$ to (3) with $\| \underline{u}-\underline{u}_0 \|_X \leq r$ where

$$r = \frac{\Delta\lambda \, \| f_0 \|}{\alpha_0} \left(\frac{1 + \sqrt{1-2H_0}}{H_0} \right) .$$

As stated above, the same proof goes through in the case of the approximate problem (5). In fact, the bounds (8) and (9) are clearly independent of h, once we show, as we do below, that the bound (6) can be chosen independent of h. Then both $\Delta\lambda$ and r will be independent of h.

We wish to generate a sequence of loads λ_i and corresponding solutions $\underline{u}_i$ and $\underline{u}_i^h$ to (3) and (5) respectively by repeated application of Lemma 1. Suppose we let $\lambda_0 = 0$. Clearly for $f = 0$ both (3) and (5) have the unique solution $\underline{u}_0 = \underline{u}_0^h = 0$. In addition, the assumption (6) is clearly satisfied since $\delta^2 S(0,\underline{\hat{v}},\underline{\bar{v}})$ is the bilinear form associated with the linear shell problem, for which the ellipticity assumption (6) has been shown to hold in [2]. Thus by Lemma 1 there is a $\lambda_1 = \lambda_0 + \Delta\lambda_1$ for which we are guaranteed unique solutions $\underline{u}_1$ and $\underline{u}_1^h$. It is immediate by writing $\delta^2 S(\underline{v},\underline{\hat{v}},\underline{\hat{v}})$ as $\delta^2 S(\underline{u}_0,\underline{\hat{v}},\underline{\hat{v}}) - (\delta^2 S(\underline{u}_0,\underline{\hat{v}},\underline{\hat{v}}) - \delta^2 S(\underline{v},\underline{\hat{v}},\underline{\hat{v}}))$ and using (9) that there exists an r' and an α' such that $\delta^2 S(\underline{v},\underline{\hat{v}},\underline{\hat{v}}) \geq \alpha' \| \underline{\hat{v}} \|^2$ for all $\underline{v}$ satisfying $\| \underline{v}-\underline{u}_0 \|_X \leq r'$, where r' depends on $\| \underline{u}_0 \|_X$. Thus by making the r in Lemma 1 somewhat smaller if necessary (by over-estimating the constant M in (9)) one can generate a sequence of loads $\lambda_i f_0$ converging to $\lambda_{cr} f$ and corresponding solutions $\underline{u}_i$ to (3) by repeated application of Lemma 1.

We show below that if the finite element approximation $\underline{u}_i^h$ to $\underline{u}_i$ satisfies

$$(10) \qquad \| \underline{u}_i - \underline{u}_i^h \|_X \leq Ch^k$$

then the finite element approximation $\underline{u}^h_{i+1}$ to $\underline{u}_{i+1}$ satisfies $\| \underline{u}_{i+1} - \underline{u}_{i+1}^h \|_X \leq C'h^k$, where both C and C' are independent of h. Using (10), we can choose the α_0 in Lemma 1 to apply to both the exact problem (3) and the approximate problem (5), thus insuring that a single $\Delta\lambda_i$ and r can be chosen in Lemma 1 for both $\underline{u}_{i+1}$ and its finite element approximation $\underline{u}^h_{i+1}$.

The method of analysis presented here parallels closely the procedure one would use to actually compute approximate solutions. To continue following the load curve beyond the critical load, however, a slightly different strategy is needed near λ_{cr}. After finding λ_{cr} with desired accuracy (when the size of the load increments that can be taken becomes small enough), instead of using the solution at the previous load step as the starting value, one instead perturbs the current solution and reduces the load.

Suppose that this starting value is $\bar{\underline{u}}^h$. Then $\bar{u}^h$ is the finite element solution for some load g, if there exists $\bar{\lambda}$ such that $\| g-\bar{\lambda}f_0\|_{L^2(\Omega)}$ is small enough, then Lemma 1 can be used to find solutions to both (3) and (5) for the load $\bar{\lambda}f_0$. This strategy may be too simplistic in more general situations, since it assumes that the direction of the load curve is known, but it seems adequate for the present analysis. A more sophisticated strategy is given in [1], when no presumption is made concerning the direction of the load curve.

We now proceed to obtain error bounds. We assume that the finite element subspaces have the following approximation properties. There is some $q \geq 3$ such that if $\underline{u} \in (H^{s-1}(\Omega) \times H^{s-1}(\Omega)) \cap (H^1_0(\Omega) \times H^1_0(\Omega))$ and $w \in H^s(\Omega) \cap H^2_0(\Omega)$, $3 \leq s \leq q$, then there is some $\tilde{\underline{u}}^h \in V_h$ such that

$$(11) \qquad \| \underline{u}-\tilde{\underline{u}}^h \|_{H^1(\Omega)\times H^1(\Omega)} \leq Ch^{s-2}$$

and there is some $\tilde{w}^h \in W_h$ such that

$$(12) \qquad \| w-\tilde{w}^h \|_{H^2(\Omega)} \leq C'h^{s-2}$$

where C and C' are independent of h.

Theorem 1. Let the finite element subpaces $V_h \subset H_0^1(\Omega)$ and $W_h \subset H_0^2(\Omega)$ satisfy (11) and (12) respectively. Suppose $\underline{u}_i$ and $\underline{u}_i^h$ are solutions to (3) and (5) respectively for load $f = \lambda_i f_0$ such that (6) is satisfied. Then there exist $\lambda_{i+1} = \lambda_i + \Delta\lambda_i$ and r such that there exist unique solutions $\underline{u}_{i+1}$ and $\underline{u}^h_{i+1}$ to (3) and (5) respectively for load $f = \lambda_{i+1} f_0$, with

$$(13) \qquad \| \underline{u}_i - \underline{u}_{i+1} \|_X \leq r, \qquad \| \underline{u}_i^h - \underline{u}_{i+1}^h \|_X < r.$$

Furthermore, if

$$(14) \qquad \| \underline{u}_i - \underline{u}_i^h \|_X \leq Ch^{s-2}$$

where C is independent of h,

$$(15) \qquad \| \underline{u}_{i+1} - \underline{u}_{i+1}^h \| \leq C'h^{s-2},$$

where C' is independent of h.

Proof: The existence and uniqueness of $\underline{u}_{i+1}$ and $\underline{u}_{i+1}^h$ satisfying (13) follows from Lemma 1 and our subsequent discussion. Let $\underline{\tilde{u}}_{i+1}^h$ and $\underline{\tilde{u}}_i^h$ satisfy (11) and (12) for $\underline{u}_{i+1}$ and $\underline{u}_i$ respectively. Let $\underline{\hat{u}}^h = \underline{u}_{i+1}^h - \underline{u}_i^h - (\underline{\tilde{u}}_{i+1}^h - \underline{\tilde{u}}_i^h)$. We have the equality

$$\delta^2 S(\underline{u}_i,\underline{\hat{u}}^h,\underline{\hat{u}}^h) + \{\delta S(\underline{u}^h_{i+1},\underline{\hat{u}}^h;\lambda_{i+1}f_0) - \delta S(\underline{u}^h_i,\underline{\hat{u}}^h;\lambda_i f_0)$$

$$- (\delta S(\underline{\tilde{u}}^h_{i+1},\underline{\hat{u}}^h;\lambda_{i+1}f_0) - \delta S(\underline{\tilde{u}}^h_i,\underline{\hat{u}}^h;\lambda_i f_0)) - \delta^2 S(\underline{u}_i,\underline{\hat{u}}^h,\underline{\hat{u}}^h)\} \tag{16}$$

$$= \delta S(\underline{u}_{i+1}\underline{\hat{u}}^h;\lambda_{i+1}f_0) - \delta S(\underline{u}_i,\underline{\hat{u}}^h;\lambda_{i+1}f_0 - (\delta S(\tilde{u}^h_{i+1},\underline{\hat{u}}^h;\lambda_{i+1}f_0) - \delta S(\tilde{u}^h,\underline{\hat{u}}^h;\lambda_i f_0))$$

The right side of (16) can be bounded by

$$Kh^{s-2} \,\| \hat{u}^h \|_X \,. \tag{17}$$

We show that the left side of (16) can be bounded from below by

$$(\alpha_0-\beta r-\beta' r^2) \,\| \underline{\hat{u}}^h \|^2_X \tag{18}$$

where β and β' are constants independent of α_0, h, and r, and α_0 is the constant in the inequality (6). As mentioned before, r can be taken small enough that $\alpha_0-\beta r-\beta' r^2 > 0$. Once we have established the inequality (18), the bound (15) follows from the triangle inequality and (14).

The appearance of $-\delta^2 S(\underline{u}^h_i,\underline{u}^h,\underline{u}^h)$ in the brackets on the left side of (16) serves to cancel out all of the linear terms in the previous four terms. There are three nonlinear terms to be bounded, corresponding to the three nonlinear terms in (3). Corresponding to the first, we have

$$\int_\Omega \left\{ \tfrac{1}{2}\left[\frac{\partial(w^h_{i+1}-w^h_i)}{\partial x_\beta} \frac{\partial(w^h_{i+1}+w^h_i)}{\partial x_\alpha} - \frac{\partial(\tilde{w}^h_{i+1}-\tilde{w}^h_i)}{\partial x_\beta} \frac{\partial(\tilde{w}^h_{i+1}+\tilde{w}^h_i)}{\partial x_\alpha} \right] - \frac{\partial w^h_i}{\partial x_\alpha} \frac{\partial \hat{w}^h}{\partial x_\beta} \right\} \cdot \left(\frac{\partial \hat{u}_j}{\partial x_\delta} + k_{\gamma\delta}\hat{w}^h \right)$$

(19)

$$= \int_\Omega \tfrac{1}{2} \frac{\partial \hat{w}^h}{\partial x_\beta} \left(\frac{\partial(w^h_{i+1}-w^h_i)}{\partial x_\alpha} + \frac{\partial(\tilde{w}^h_{i+1}-\tilde{w}^h_i)}{\partial x_\alpha} \right) \cdot \left(\frac{\partial \hat{u}^h_\gamma}{\partial x_\delta} + k_{\gamma\delta}\hat{w}^h \right)$$

$$+ \left(\frac{\partial(\tilde{w}^h_{i+1}-\tilde{w}^h)}{\partial x_\beta} \frac{\partial(\tilde{w}^h_i-w^h_i)}{\partial x_\alpha} \right) \cdot \left(\frac{\partial \hat{u}_\gamma}{\partial x_\delta} + k_{\gamma\delta}\hat{w}^h \right) .$$

Using the embedding $L^4(\Omega) \subset H^1(\Omega)$, the first term on the right side of (19) can be bounded by $Kr\| \underline{\hat{u}}^h \|^2_X$, while the second term on the right hand side of (19) can be bounded by $K'h^{s-2}\| \underline{\hat{u}}^h\|_X$, and moved over to the right side of (16). The second nonlinear term in (3) can be handled in the same way as (19). Corresponding to the last nonlinear term in (3), we have

$$\int_\Omega \left\{ \frac{1}{3} \left[\frac{\partial(w_{i+1}^h - w_i^h)}{\partial x_\alpha} \left(\frac{\partial w_{i+1}^h}{\partial x_\beta} \frac{\partial w_{i+1}^h}{\partial x_\gamma} + \tfrac{1}{2} \left(\frac{\partial w_{i+1}^h}{\partial x_\beta} \frac{\partial w_i^h}{\partial x_\gamma} + \frac{\partial w_{i+1}^h}{\partial x_\gamma} \frac{\partial w_i^h}{\partial x_\beta} \right) + \frac{\partial w_i^h}{\partial x_\beta} \frac{\partial w_i^h}{\partial x_\gamma} \right) \right. \right.$$

$$- \frac{\partial(\tilde{w}_{i+1}^h - \tilde{w}_i^h)}{\partial x_\alpha} \left(\frac{\partial \tilde{w}_{i+1}^h}{\partial x_\beta} \frac{\partial \tilde{w}_{i+1}^h}{\partial x_\gamma} + \tfrac{1}{2} \left(\frac{\partial \tilde{w}_{i+1}^h}{\partial x_\beta} \frac{\partial \tilde{w}_i^h}{\partial x_\gamma} + \frac{\partial \tilde{w}_{i+1}^h}{\partial x_\gamma} \frac{\partial \tilde{w}_i^h}{\partial x_\beta} \right) + \frac{\partial \tilde{w}_i^h}{\partial x_\beta} \frac{\partial \tilde{w}_i^h}{\partial x_\gamma} \right) \Bigg]$$

$$\left. - \frac{\partial w_i^h}{\partial x_\beta} \frac{\partial w_i^h}{\partial x_\gamma} \frac{\partial \hat{w}^h}{\partial x_\alpha} \right\} \frac{\partial \hat{w}^h}{\partial x_\delta}$$

$$= \int_\Omega \frac{1}{3} \left\{ \frac{\partial \hat{w}^h}{\partial x_\alpha} \left(\frac{\partial(w_{i+1}^h - w_i^h)}{\partial x_\beta} \frac{\partial(w_{i+1}^h - w_i^h)}{\partial x_\gamma} \right. \right.$$

(20)

$$+ \tfrac{1}{2} \left(\frac{\partial(w_{i+1}^h - w_i^h)}{\partial x_\beta} \frac{\partial(\tilde{w}_{i+1}^h - \tilde{w}_i^h)}{\partial x_\gamma} + \frac{\partial(w_{i+1}^h - w_i^h)}{\partial x_\gamma} \frac{\partial(\tilde{w}_{i+1}^h - w_i^h)}{\partial x_\beta} \right)$$

$$\left. + \frac{\partial(\tilde{w}_{i+1}^h - \tilde{w}_i^h)}{\partial x_\beta} \frac{\partial(\tilde{w}_{i+1}^h - \tilde{w}_i^h)}{\partial x_\gamma} \right) + \frac{3}{2} \left[\frac{\partial \hat{w}^h}{\partial x_\alpha} \left(\frac{\partial(w_{i+1}^h - w_i^h)}{\partial x_\beta} + \frac{\partial(\tilde{w}_{i+1}^h - \tilde{w}_i^h)}{\partial x_\beta} \right) \frac{\partial w_i^h}{\partial x_\gamma} \right.$$

$$+ \frac{\partial(\tilde{w}_{i+1}^h - \tilde{w}_i^h)}{\partial x_\alpha} \frac{\partial(\tilde{w}_{i+1}^h - \tilde{w}_i^h)}{\partial x_\beta} \frac{\partial(\tilde{w}_i^h - \tilde{w}_i^h)}{\partial x_\gamma}$$

$$\left. \left. + \frac{\partial(\tilde{w}_{i+1}^h - \tilde{w}_i^h)}{\partial x_\alpha} \frac{\partial(\tilde{w}_{i+1}^h - w_i^h)}{\partial x_\beta} \frac{\partial(\tilde{w}_i^h + w_i^h)}{\partial x_\gamma} \right] \right\} \frac{\partial \hat{w}^h}{\partial x_\delta}$$

It is clear that the right side of (20) can be bounded similarly to (19) except that the constants will depend on $\| w_i^h \|_{H^2(\Omega)}$, as in our proof of existence and uniqueness.

Near the critical load, we still obtain optimal error bounds if we replace $\underline{u}_i^h$ by $\bar{\underline{u}}^h$ as above and assume the bound

$$\| \bar{\underline{u}} - \bar{\underline{u}}^h \| \le Ch^{s-2} \quad ,$$

where $\bar{\underline{u}}$ is the exact solution to (3) for the load g. Such an assumption is justifiable if the displacement $\bar{\underline{u}}$ occurs prior to the critical load for the load distribution g.

III. Mixed Formulation

Let $(L^2(\Omega))^3 = L^2(\Omega) \times L^2(\Omega) \times L^2(\Omega)$ with norm

$$\| \underline{v} \|_{(L^2(\Omega))^3} = (\| v_j \|^2_{L^2(\Omega)})^{\frac{1}{2}}$$

Let $\hat{X} = X \times (L^2(\Omega))^3 \times (L^2(\Omega))^3$. On $\hat{X}$ we define the norm

$$\| \underline{\phi} \|_{\hat{X}} = (\| \underline{u} \|^2_X + \| \underline{N} \|^2_{(L^2(\Omega))^3} + \| \underline{M} \|^2_{(L^2(\Omega))^3})^{\frac{1}{2}} ,$$

where $\underline{\phi} = (\underline{u}, \underline{N}, \underline{M})$, with $\underline{N}$, $\underline{M}$, 2×2 symmetric tensors, which represent the direct stresses and the bending moments respectively. The mixed formulation of the shell problem may be expressed as: find $\underline{\phi} = (\underline{u}, \underline{N}, \underline{M}) \in \hat{X}$ such that

$$\begin{aligned} R(\underline{\phi};f) &= \min_{\underline{\psi} \in X} R(\underline{\psi};f) \\ &= \int_\Omega \Big[-\tfrac{1}{2} A_{\alpha\beta\gamma\delta} N_{\alpha\beta} N_{\gamma\delta} - \tfrac{1}{2} G_{\alpha\beta\gamma\delta} M_{\alpha\beta} M_{\gamma\delta} \\ &\quad + \left(\frac{\partial u_\beta}{\partial x_\alpha} + k_{\alpha\beta} w + \tfrac{1}{2} \frac{\partial w}{\partial x_\alpha} \frac{\partial w}{\partial x_\beta} \right) N_{\alpha\beta} \\ &\quad - \frac{\partial^2 w}{\partial x_\alpha \partial x_\beta} M_{\alpha\beta} - \underline{f}\,\underline{u} \Big] , \end{aligned} \tag{21}$$

or equivalently as: find $\phi \in \hat{X}$ such that

$$\delta R(\underline{\phi},\underline{\bar{\phi}};f) = \int_\Omega - A_{\alpha\beta\gamma\delta}N_{\alpha\beta}\bar{N}_{\gamma\delta} - G_{\alpha\beta\gamma\delta}M_{\alpha\beta}\bar{M}_{\gamma\delta}$$

$$+ \left(\frac{\partial u_\beta}{\partial x_\alpha} + k_{\alpha\beta}w\right)\bar{N}_{\alpha\beta} + N_{\alpha\beta}\left(\frac{\partial \bar{u}_\beta}{\partial x_\alpha} + k_{\alpha\beta}\bar{w}\right)$$

$$(22) \qquad + \tfrac{1}{2}\frac{\partial w}{\partial x_\alpha}\frac{\partial w}{\partial x_\beta}\bar{N}_{\alpha\beta} + \tfrac{1}{2}N_{\alpha\beta}\left(\frac{\partial w}{\partial x_\alpha}\frac{\partial \bar{w}}{\partial x_\beta} + \frac{\partial w}{\partial x_\beta}\frac{\partial \bar{w}}{\partial x_\alpha}\right)$$

$$- \frac{\partial^2 w}{\partial x_\alpha \partial x_\beta}\bar{M}_{\alpha\beta} - M_{\alpha\beta}\frac{\partial^2 \bar{w}}{\partial x_\alpha \partial x_\beta} - \underline{f}\,\underline{u} = 0 ,$$

$$\text{all } \underline{\bar{\phi}} = (\underline{\bar{u}},\underline{\bar{N}},\underline{\bar{M}}) \in \hat{X} .$$

For the finite element approximation to the solution $\underline{\phi}$ of (22), choose finite dimensional subspaces $S_N^h \subset (L^2(\Omega))^3$ and $S_M^h \subset (L^2(\Omega))^3$, along with V_h and W_h of the previous section. Let $\hat{X}_h = \hat{X}_h \times S_N^h \times S_M^h$ and find $\underline{\phi}^h = (u^h,\underline{N}^h,\underline{M}^h) \in \hat{X}_h$ such that

$$(23) \qquad R(\underline{\phi}^h;f) = \min_{\underline{\phi}^h \ \hat{X}_h} R(\underline{\phi}^h;f) ,$$

or equivalently : find $\underline{\phi}^h \in \hat{X}^h$ such that

$$(24) \qquad \delta R(\underline{\phi}^h,\underline{\bar{\phi}}^h;f) = 0 . \qquad \text{all } \underline{\phi}^h \in \hat{X}_h .$$

We assume the S_N^h and S_M^h have the following approximation properties. If $\underline{N} \in (H^{s-2}(\Omega))^2$, $\underline{M} \in (H^{s-2}(\Omega))^2$, then there is some $\underline{\tilde{N}}^h \subset S_N^h$, $\underline{\tilde{M}}^h \subset S_M^h$ such that

(25) $$\| \underline{N}-\underline{\tilde{N}}^h \|_{(L^2(\Omega))^3} \le Ch^{s-2} \quad ,$$

(26) $$\| \underline{M}-\underline{\tilde{M}}^h \|_{(L^2(\Omega))^3} \le Ch^{s-2}$$

The same analysis of existence and uniqueness goes through for the mixed method as for the minimum energy method in the previous section, once we prove the following lemma. Let P_N denote the orthogonal projection of $(L^2(\Omega))^3$ into S_N^h.

Lemma 2. Suppose that the subspaces S_N^h and S_M^h satisfy the following: all vectors $(\partial u_1^h/\partial x_1, \partial u_2^h/\partial x_2, \frac{1}{2}(\partial u_1^h/\partial x_1 + \partial u_2^h/\partial x_2))$ are contained in S_N^h for all $(u_1, u_2) \in V^h$ and all vectors $(w_{xx}^h, w_{yy}^h, w_{xy}^h)$ are contained in S_M^h for all $w^h \in W^h$. Then

(27) $$\sup_{\underline{\hat{\phi}} \in X} \delta^2 R(0, \underline{\phi}, \underline{\hat{\phi}}) \ge \alpha_0 \| \phi \|_{\hat{X}} \| \underline{\hat{\phi}} \|_{\hat{X}} \quad , \quad \text{all } \underline{\phi} \in \hat{X} ,$$

and for h small enough

(28) $$\sup_{\underline{\hat{\phi}}^h \in \hat{X}_h} \delta^2 R(0, \underline{\phi}^h, \underline{\hat{\phi}}^h) \ge \alpha_0 \| \underline{\phi}^h \|_{\hat{X}} \| \underline{\phi}^h \|_{\hat{X}} \quad , \quad \text{all } \underline{\phi}^h \in \hat{X}_h ,$$

where α_0 is a constant independent of h.

Proof. Since the supremum in (29) is only taken over $\hat{X}_h$, (26) does not follow from (27). In fact, it is necessary to restrict the subspaces S_N^h and S_M^h in order to prove (28). We shall prove (28). It will be clear that (27) can be proved in the same manner.

We take $\underline{N}$ to be the vector (N_{11},N_{22},N_{12}) and $\underline{M}$ to be the vector (M_{11},M_{22},M_{12}). $\delta^2R(0,\underline{\phi},\hat{\underline{\phi}})$ can be expressed in matrix form as

$$\begin{aligned}\delta^2R(0,\hat{\underline{\phi}}^h,\hat{\underline{\phi}}^h) &= -(\underline{N}^h,A\underline{N}^h) - (\underline{M}^h,G\bar{\underline{M}}^h) + (T_1\underline{u}^h,\hat{\underline{N}}^h) \\ &\quad + (\underline{N}^h,T_1\hat{\underline{u}}^h) - (T_2w^h,\hat{\underline{M}}^h) - (\underline{M}^h,T_2\hat{w}^h)\end{aligned} \tag{29}$$

where $T_1\underline{u}^h$ has components $\partial u^h_\beta/\partial x_\alpha + k_{\alpha\beta}w^h$ and T_2w^h is the vector $(w^h_{xx},w^h_{yy},w^h_{xy})$. Both the matrices A and G are positive definite. Now choose

$$\hat{\underline{N}}^h = -\underline{N}^h + A^{-1}P_N(T_1\underline{u}^h) \quad , \qquad \hat{\underline{u}}^h = 2\underline{u}^h \quad ,$$

$$\hat{\underline{M}}^h = -M^h - G^{-1}(T_2w^h) \quad .$$

It is clear that $\hat{\underline{N}}^h$ and $\hat{\underline{M}}^h$ are contained in S^h_N and S^h_M respectively, and that there exists a constant C such that

$$\| \hat{\underline{\phi}}^h \|_{\hat{X}} \leq C \| \underline{\phi}^h \|_{\hat{X}} \quad .$$

We obtain

$$\begin{aligned}&(\underline{N}^h,A\underline{N}^h) + (M^h, G\underline{M}^h) + (T_1\underline{u}^h, A^{-1}T_1\underline{u}^h) \\ &\quad + (T_2w^h,G^{-1}T_2w^h) + (T_1\underline{u}^h,A^{-1}(P_N(k_{\alpha\beta}w^h)-k_\alpha w^h)) \\ &\geq (C'-K'h^2)\ \| \hat{\underline{\phi}} \|\ \| \underline{\phi} \| \ .\end{aligned} \tag{30}$$

Here we have used the approximation property of S^h_N and the ellipticity property

$$\| T_1\underline{u} \|_X \geq K \| \underline{u} \|_X \quad ,$$

proved in [2]. This establishes (28). The inequality (27) can be proved in the same way.

Theorem 2. Let the finite element subspaces $V_h \subset H_0^1(\Omega)$ and $W_h \subset H_0^2(\Omega)$ satisfy (11) and (12) respectively. Let the subspaces $S_N^h \subset (L^2(\Omega))^3$ and $S_M^h \subset (L^2(\Omega))^3$ satisfy (25) and (26) respectively along with the hypotheses of Lemma 2. If $\underline{\phi}_i$ and $\underline{\phi}_i^h$ are solutions to (22) and (24) respectively for the load $f = \lambda_i f_0$ such that (27) and (28) are satisfied, then there exist $\lambda_{i+1} = \lambda_i + \Delta\lambda_i$ and r such that there exist unique solutions $\underline{\phi}_{i+1}$ and $\underline{\phi}_{i+1}^h$ to (22) and (24) respectively for the load $f = \lambda_{i+1} f_0$, with

(31) $$\|\underline{\phi}_i - \underline{\phi}_{i+1}\|_{\hat{X}} \leq r \quad , \quad \|\underline{\phi}_i^h - \underline{\phi}_{i+1}^h\|_{\hat{X}} \leq r \quad .$$

Furthermore, if

(32) $$\|\underline{\phi}_i - \underline{\phi}_i^h\|_{\hat{X}} \leq K h^{s-2} \quad ,$$

where K is independent of h, then

(33) $$\|\underline{\phi}_{i+1} - \underline{\phi}_{i+1}^h\|_{\hat{X}} \leq K' h^{s-2} \quad ,$$

where K' is independent of h.

Proof. The proof of existence and uniqueness of $\underline{\phi}_{i+1}$ and $\underline{\phi}_{i+1}^h$ satisfying (31) follows in the same way as for the minimum energy formulation except that the constant M in (9), and thus also r, is independent of $\|w\|_{H^2(\Omega)}$. The bound (33) can be established in the same manner as in the previous section. For the mixed method all estimates on the nonlinear terms are similar to (19). No estimates like (20) enter.

IV. Concluding Remarks

If one assumes Kirchhoff's hypothesis, the weak form of the shell equations involves second partials of the transverse displacement w. Thus C^1

finite elements must be used to approximate w. For linear problems, a major reason for considering mixed methods is to allow, by integration by parts, the use of C^0-finite elements, as in [6] and [8]. In this paper we did not consider this possibility because results of the type obtained here are not straight-forward to obtain. One difficulty is that the system is no longer quasi-linear, and the bound (9) does not hold. Thus Kantorovich's theorem can not be applied in a straight-forward way. If one does not assume Kirchhoff's hypothesis, one also has only first partials of w appearing, but with additional variables, the transverse shear stresses Q_α and the rotations ϕ_α. This system is also not quasi-linear.

It is the author's opinion that the fact that the size of the load steps that can be taken is independent of the size of w for the mixed method but not for the minimum potential energy method is significant. One does, however have a system with many more independent variables which is not positive definite. When this larger system has been solved, however, one has the stresses and bending moments explicitly, and doesn't need to obtain them by differentiation of the displacements, a non-negligible cost.

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